The *Ex Ante* Likelihood Of Bubbles

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Abstract

Limits to arbitrage explain how a speculative bubble can be sustained. They do not explain how often a bubble will occur. To do that, you need an on/off switch which sporadically amplifies speculator biases, causing arbitrageur constraints to bind and a bubble to form. I propose such an on/off switch based on social interactions between speculators. Bubbles should occur more often in assets where increases in past returns make excited speculators relatively more persuasive to their peers. I verify this *ex ante* prediction about bubble likelihoods and show it is robust to *ex post* disagreement about bubble definitions.

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1 Introduction

For all the attention they receive, one of the most striking things about speculative bubbles is how rare they are. For example, Kindleberger (1978) examined national price indexes and found only 34 bubble episodes in over 400 years! What determines the \textit{ex ante} likelihood of a speculative bubble for a given asset?

Policymakers want an answer to this ‘how often’ question. Economists regularly get asked questions like: “By driving rates ever lower”, is the “Fed helping to fuel dangerous bubbles in stocks or other risky assets”?\footnote{“Fed Set To Cut Rates For First Time In Decade. Is It A Risk?” \textit{AP News}. 7/29/2019.} When Alan Greenspan expressed concerns about “froth in the housing market”\footnote{“Greenspan Is Concerned About ‘Froth’ In Housing.” \textit{The New York Times}. 5/21/2005.}, he was explicitly talking about the likelihood of a future bubble.

To be sure, the limits-to-arbitrage framework (Shleifer and Vishny, 1997) explains how a speculative bubble can be sustained. If arbitrageurs are handcuffed by some sort of constraint, they may not be able to correct pricing errors caused by biased speculators. But, ‘how’ and ‘how often’ are two fundamentally different kinds of questions. And, there are two reasons why the limits of arbitrage cannot answer questions about the \textit{ex ante} likelihood of speculative bubbles on its own.

First, knowing about biases and constraints is not the same thing as knowing how often they will bind. Researchers have tabulated long lists of the various arbitrageur constraints—e.g., short-sale bans (Miller, 1977), margin requirements (Gromb and Vayanos, 2002), bounded rationality (Gabaix, 2014)—and psychological biases—e.g., overconfidence (Daniel et al., 1998; Scheinkman and Xiong, 2003), heuristic updating (Hong and Stein, 1999; Barberis et al., 2015; Bordalo et al., 2018), sentiment (Baker and Wurgler, 2006)—needed to sustain a speculative bubble. By the logic of limits-to-arbitrage, any bias×constraint pair drawn from these two lists has the potential to combine at any moment to produce a bubble.

Second, while any given limits-to-arbitrage model must point to a specific bias×constraint pair when justifying a particular speculative bubble, different bubble episodes often appear to be driven by different psychological biases and arbitrageur constraints. In fact, over the last 30 years, you could argue that there have been as many limits-to-arbitrage models as bubble episodes. Thus, any
explanation for the *ex ante* likelihood of bubbles should not be tied to a specific limits-to-arbitrage story. Instead, it should explain how often you should expect that some psychological bias will cause some arbitrageur constraint to bind.

This paper proposes and then tests a first such explanation for the *ex ante* likelihood of bubbles. This explanation is based on the idea that speculators “go mad in herds [but] only recover their senses slowly and one by one (Mackay, 1841)”. Other researchers have pointed out that social interactions between speculators might distort asset prices (Shiller, 1984; Shive, 2010; Burnside et al., 2016; Hirshleifer, 2020). However, this paper is applying the idea in a fundamentally different way. Earlier papers treated social interactions as yet another reason why speculators might be biased. By contrast, in this paper, social interactions between speculators serve as an on/off switch, regulating the size of the excited-speculator population in the market independent of what biases they suffer from.

I begin the analysis in Section 2 with an economic model. The model contains two parts. First, there is a completely standard limits-to-arbitrage model which determines equilibrium prices conditional on the number of excited speculators in the market. Then, there is a time-varying population of excited speculators whose size is determined by social interactions. During normal times, these social interactions cause the excited-speculator population to dwindle away to zero, leaving prices unaffected by any psychological biases these speculators might have. However, under certain conditions, the very same social interactions produce exponential growth in the excited-speculator population. When that happens, their psychological biases (whatever they happen to be) cause arbitrageurs’ constraints (whatever they happen to be) to bind and a bubble to form.

The model predicts that bubbles should occur more often in assets where small increases in past returns make excited speculators relatively more persuasive to their peers. In Section 3, I empirically verify this *ex ante* prediction about bubble likelihoods using data on industry-level stock returns. To be sure, there is no general consensus about which market episodes were bubbles *ex post*, and I will not settle this debate here. However, a key takeaway from this paper is that it is possible to test *ex ante* predictions about bubble likelihoods even in the presence of this *ex post* disagreement about bubble definitions. By analogy, it
is universally recognized that suicide is the most likely form of gun death in America even though coroners sometimes disagree after the fact about whether a particular gun death was a suicide, a homicide, or an accident.\(^3\)

This paper borrows from and builds on several strands of related literature. First, there is the limit-to-arbitrage literature (Shleifer and Summers, 1990; Shleifer and Vishny, 1997; Barberis and Thaler, 2003; Gromb and Vayanos, 2010), which provides an explanation for how equilibrium pricing errors can exist. The starting point for this paper is that the limits of arbitrage cannot explain how often such equilibrium pricing errors should occur. Empirical papers such as Ofek and Richardson (2001), Brunnermeier and Nagel (2004), and Xiong and Yu (2011) have been influential in showing that speculative bubbles can occur and that feedback trading (Cutler et al., 1990; De Long et al., 1990) takes place during bubble episodes. Recent research has studied non-price features of bubbles (DeFusco et al., 2017; Glaeser and Nathanson, 2017; Barberis, 2018; An et al., 2019; Haddad et al., 2019; Liao and Peng, 2019). Finally, as a microfoundation for the social interactions, this paper points to a literature on peer effects (Shiller and Pound, 1989; Hong et al., 2004; Kaustia and Knüpfer, 2012; Bursztyn et al., 2014; Bailey et al., 2018).

2 Economic Model

This section presents an economic model that makes \textit{ex ante} predictions about the likelihood of a future speculative bubble. I start in Subsection 2.1 by writing down a standard limits-to-arbitrage model based on Hong and Stein (1999). This model pins down equilibrium prices as a function of the number of excited speculators in the market. However, the specific details of the model are not critical. Any similar model will do, and to illustrate this point I show that, while changing the parameters of the limits-to-arbitrage model will affect the evolution of any speculative bubble once it gets started, such changes will not affect the \textit{ex ante} likelihood of a speculative bubble taking place.

Then, in Subsection 2.2, I describe the mechanism determining whether there are any excited speculators in the market. This mechanism is based on

\(^3\)“Gun Deaths In America.” FiveThirtyEight. 6/6/2016.
the old adage that speculators “go mad in herds [but] only recover their senses slowly and one by one (Mackay, 1841)”. Most of the time, social interactions cause this population to dwindle away to nothing, leaving prices unaffected by any psychological biases these speculators might have. However, under certain conditions, the very same social interactions cause the excited-speculator population to explode. When that happens, speculator biases (whatever those biases happen to be) cause arbitrageur constraints (whatever those constraints happen to be) to bind and a speculative bubble forms.

Finally, in Subsection 2.3, I bring these two separate pieces together to show that researchers can use a single novel parameter, \( \theta \), to predict the likelihood of future bubble episodes. \( \theta \) captures how much more persuasive speculators are to their peers when an asset has experienced high past returns. When house prices rise by 10%, it is national news. Everyone starts talking about housing. By contrast, while a 10% increase in the price of textile stocks is a big deal for market participants, it is not going to allow these participants to recruit many of their friends. Thus, given the same initial conditions, housing should be more likely to experience a speculative bubble than textiles regardless of the specific speculator biases and arbitrageur constraints involved.

### 2.1 Limits to Arbitrage

Here is the limits-to-arbitrage model that determines equilibrium prices conditional on the size of the excited-speculator population.

Time proceeds in discrete steps indexed by \( t = 1, 2, \ldots \). There is a single risky asset with a per-period payout of \( v_t \) dollars per share:

\[
\Delta v_t = \kappa_v \cdot (\mu_v - v_{t-1}) + \sigma_v \cdot \varepsilon_{v,t} \tag{1}
\]

\( \mu_v \gg 0 \) is the risky asset’s average payout per period, \( \kappa_v \in (0, 1) \) is its mean-reversion coefficient, \( \sigma_v > 0 \) is the volatility of changes in its per-period payout, and \( \varepsilon_{v,t} \sim N(0, 1) \) is an IID shock. \( \psi \geq 0 \) is the supply of risky shares.

The market contains two kinds of agents: newswatchers and speculators. Newswatchers incorporate news about an asset’s fundamental value into the price. Every period there is a new unit mass of newswatchers indexed by \( j \in [0, 1] \). The \( j \)th newswatcher in period \( t \) chooses his demand, \( x_{j,t} \), to maximize his expected
end-of-period utility from consuming the asset’s time-$t$ payout given the price $p_t$:

$$x_{j,t} = \arg \max_x E_j \left[ -e^{-\gamma(v_t-p_t)x} \right]$$

(2)

$\gamma > 0$ is the risk-aversion coefficient.

The newswatchers within each cohort have heterogeneous beliefs. Prior to the start of period $t$, the $j$th newswatcher observes a private signal, $s_{j,t}$, about the asset’s time-$t$ payout where:

$$v_t = s_{j,t} + \varepsilon_{j,t}$$

(3)

$s_{j,t}$ denotes the $j$th newswatcher’s beliefs after observing his private signal and $\varepsilon_{j,t} \sim \text{N}(0,1)$ is noise.

There are two additional assumptions mirroring Admati (1985). First, the heterogeneous beliefs of each newswatcher cohort are correct on average:

$$v_t = E[s_{j,t}] = \int_0^1 s_{j,t} \cdot dj$$

(4)

Second, although different newswatchers get different private signals, every newswatcher’s private signal has unit precision, $\text{Var}[\varepsilon_{j,t}] = 1$.

Newswatchers are the constrained arbitrageurs in the model. Their constraint takes the form of bounded rationality.\(^4\) “First, at every time $t$, newswatchers formulate their asset demands based on the static-optimization notion that they buy and hold... Second, and more critically... [newswatchers] do not condition on current or past prices. In other words, the equilibrium concept is a Walrasian equilibrium with private valuations, as opposed to a fully revealing rational expectations equilibrium. (Hong and Stein, 1999)”

This choice of constraint is convenient because it implies that newswatcher beliefs about future realizations of $n_t$ do not affect an asset’s current price level. This independence completely separates the limits of arbitrage (how a speculative bubble is sustained) from the madness of crowds (how often they occur). In essence, it is like using log utility in a macro-finance model when things like hedging demand are not germane to the model’s conclusions.

\(^{4}\text{You could model newswatchers who were short-sale constrained rather than myopic. In such a model, the option to sell at an inflated price during future bubble episodes would push up an asset’s price today as in Scheinkman and Xiong (2003). If the goal was to understand cross-sectional differences in the average price levels, it would be exactly the right model to study. But, this is not the goal. We want to predict the ex ante likelihood of future bubbles.}$$
The market contains \( n_t \in [0, 1) \) excited speculators. For now, take the number of excited speculators as given. The role of excited speculators is to cause an asset’s price to move for non-fundamental reasons. They do so whenever \( n_t > 0 \). The aggregate demand coming from the crowd of excited speculators, \( Z_t(n) \), is proportional to the asset’s recent performance:

\[
Z_t(n) = (\lambda \cdot r_{t-1}) \times n
\]  

\( \lambda > 0 \) is the strength of excited speculators’ bias. A larger \( \lambda \) means excited speculators are more biased. \( r_{t-1} \overset{\text{def}}{=} (p_{t-1} + v_{t-1})/p_{t-2} \) is the past realized return.

The aggregate demand coming from newswatchers and the population of excited speculators must clear the market:

\[
\int_{0}^{1} x_{j,t} \cdot dj + Z_t(n) = \psi
\]  

Newswatchers ignore the information content of prices, so they have demand:

\[
x_{j,t} = (s_{j,t} - p_t)/\gamma
\]  

If the \( j \)th newswatcher’s private signal results in beliefs that are higher than the price, \( (s_{j,t} - p_t) > 0 \), then he will buy; otherwise, he will sell.

**Proposition 2.1 (Limits to Arbitrage).** When there are \( n_t = n \) excited speculators in the market, equilibrium asset prices are given by:

\[
P_t(n) = v_t - \gamma \times \psi + \gamma \cdot (\lambda \cdot r_{t-1}) \times n
\]  

This price is increasing in the fundamental value, \( v_t \); decreasing in the number of shares, \( \psi \); and, increasing in the number of excited speculators, \( n \in [0, 1) \).

### 2.2 On/Off Switch

Here is how I model the madness-of-crowds mechanism, which controls whether or not there are any excited speculators in the market.

There are \( K \gg 1 \) speculators. Let \( N_{\tau} \geq 0 \) denote the number of speculators excited about a particular asset, so \( n_{\tau} \overset{\text{def}}{=} N_{\tau}/K \) denotes the excited fraction.\(^5\) In this paper, social interactions between speculators are not a psychological bias in and of themselves. Instead, they are operating as an on/off switch, regulating whether some psychological bias will cause some arbitrageur constraint to bind.

\(^5\)It is possible to extend the madness-of-crowds mechanism to a multi-asset setting, \( k = 1, \ldots, K \), by thinking about \( N_{k,\tau} \) as the number of speculators currently excited about the \( k \)th asset.
To make this distinction as clear as possible, I set the model up so that the limits of arbitrage and this on/off switch operate on completely different time scales. The fraction of excited speculators at time \( t \), \( n_t \in [0, 1) \), will be the steady-state outcome of social interactions that play out in continuous time, \( \tau \geq 0 \).

Popular accounts of bubble formation typically describe bubbles as “social epidemics” where “news of price increases spurs investor enthusiasm which spreads by psychological contagion from person to person, in the process amplifying stories that might justify the price increases and bringing in a larger and larger class of investors, who, despite doubts about the real value of an investment, are drawn to it partly through envy of others’ successes and partly through a gambler’s excitement. (Shiller, 2000)” There is also a large academic literature showing that traders are more likely to enter a market if they already know someone else who has done so (Shiller and Pound, 1989; Hong et al., 2004; Kaustia and Knüpfer, 2012; Bursztyn et al., 2014).

I use the following rule to capture this notion of positive feedback between past returns and speculator excitement:

\[
\lim_{\Delta \tau \searrow 0} \Pr \left[ n_{\tau+\Delta \tau} - n_{\tau} = +1/K \mid n_{\tau} = n, r \right] = \theta \cdot r \cdot (1 - n) \times n
\] (9)

Apathetic speculators get excited via interactions with their \( n \) excited friends. \( \theta \cdot r \cdot (1 - n) \) represents this per capita excitation rate. \( (1 - n) \) is the size of the apathetic speculator population. \( r \in (0, \infty) \) is an asset’s past performance. 

\( \$ \cdot r \) represents the amount of money you would have today if you had invested \( \$ \) in a particular asset several years ago. Whenever \( r \gg 1 \), apathetic speculators have missed out on a profitable investment opportunity. \( \theta \in (0, 1) \) reflects how much more persuasive an asset’s excited speculators are following an increase in past returns. In the late 1990s, tech-stock appreciation generated a lot of additional word-of-mouth buzz, \( \theta \approx 1 \); whereas, textile stocks would have to have very strong returns for excited speculators to have any sway on their friends, \( \theta \approx 0 \).

Conventional narratives also emphasize that, while speculators “go mad in herds, [they] only recover their senses slowly and one by one (Mackay, 1841)”.

This is the motivation for the second rule:

\[
\lim_{\Delta \tau \searrow 0} \Pr \left[ n_{\tau+\Delta \tau} - n_{\tau} = -1/K \mid n_{\tau} = n, r \right] = 1 \times n
\] (10)
Multiplying $n$ by the 1 equates the phrase “slowly and one by one” with a constant per capita departure rate. The rate at which each excited speculator comes to his senses is the same regardless of whether 10% or 90% of all speculators are currently excited. Moreover, because Equation (10) does not contain $r$, this rate is also independent of the asset’s past return.\(^6\)

The start of speculative bubbles has been a topic of mystery. “An event that is not of unusual size or duration can trigger a sharp financial reaction. (Minsky, 1970)” The exact same shock—e.g., a change in interest rates—might trigger the madness of crowds for one asset but not another. At first glance, these irregularities are deeply “unsatisfying to [anyone] seeking scientific certitude (Shiller, 2000)”. This paper uses the fact that investors only slowly recover their senses as an explanation for these irregularities. “Slowly and one by one” is a way of taking “the strangeness of collective behavior out of the heads of individual actors and putting it into the dynamics of situations (Granovetter, 1978)”.

There is nothing about these two rules that suggests a sudden qualitative change in the excited-speculator population as an asset’s past returns change. Both rules are smooth and continuous functions, so you might expect that a slight increase in an asset’s past return would always result in a slight increase in its excited-speculator population. But, this is not what happens. The two rules actually produce a sudden change in steady-state behavior as an asset’s past returns cross a critical threshold.\(^7\)

Let $n_\tau(n_0, \theta, r)$ be the fraction of speculators excited about an asset at time $\tau \geq 0$ if $n_0$ were excited at time $\tau = 0$:\(^8\)

$$n_\tau(n_0, \theta, r) \overset{\text{def}}{=} \left\{ n \in [0, 1) : n = \int_0^\tau [\theta \cdot r \cdot (1 - n_u) - 1] \cdot n_u \cdot du \right\} \quad (11)$$

\(^6\)The choice of $1 \times n$ rather than $\omega \times n$ for $\omega > 0$ is without loss of generality (see Online Appendix). The key assumption is that the rate at which excited speculators calm down must be less sensitive to changes in the size of the excited-speculator crowd, $n$, and the asset’s past returns, $r$, than the rate at which apathetic speculators get excited. The functional forms in Equations (9) and (10) are the simplest way to model this assumption. It is also possible to incorporate stochastic fluctuations in these population dynamics (see Online Appendix).

\(^7\)This large qualitative change in the steady-state behavior is known as a “bifurcation” (Hirsch et al., 2012; Guckenheimer and Holmes, 2013; Kuznetsov, 2013; Strogatz, 2014).

\(^8\)Standard texts (Arnol’d, 2012) show that $n_\tau(n_0, \theta, r)$ is unique for all $\tau \geq 0$ and $n_0 \in [0, 1)$ because $\theta \cdot r \cdot (1 - n) \cdot n - n$ is continuously differentiable on an open interval containing $[0, 1)$. 
A steady-state population, $\bar{n} \in [0,1)$, is a population such that

$$SS(\theta,r) \overset{\text{def}}{=} \left\{ \bar{n} \in [0,1) : n_\tau(\bar{n},\theta,r) = \bar{n} \ \forall \tau \geq 0 \right\}$$

(12)

We say that a particular steady-state value, $\bar{n} \in SS(\theta,r)$, is stable if small perturbations away from $\bar{n}$ die out over time.\(^9\) The proposition below analytically characterizes the critical return threshold, $r_*$, such that there will be a non-zero steady-state population of excited speculators in the market whenever $r > r_*$.\(^{9,1}\)

**Proposition 2.2** *(On/Off Switch)*. Define $r_* \overset{\text{def}}{=} 1/\theta$.

1. If $r < r_*$, there is only one steady-state value for the excited-speculator population, $SS(\theta,r) = \{0\}$. This lone steady state, $\bar{n} = 0$, is stable.

2. If $r > r_*$, there are two steady-state values, $SS(\theta,r) = \{0,(r-r_*)/r > 0\}$.

   Only the strictly positive steady state, $\bar{n} = (r-r_*)/r > 0$, is stable.

When $r < r_*$, any initial population of excited speculators, $n_0 > 0$, quickly loses interest. But, as soon as an asset’s return crosses the critical threshold, $r > r_*$, that same initial population gives rise to a persistent crowd of excited speculators.

To see why social interactions between speculators can serve as an on/off switch, consider what happens when there is only one speculator excited about an asset, $N_\tau = n_\tau \cdot K = 1$. In this situation, the entire population of excited speculators will go extinct if its lone member cannot excite at least one of his apathetic friends before he himself comes to his senses:

$$\Pr[\Delta N_\tau = +1 \mid N_\tau = 1] \approx \Pr[\Delta N_\tau = -1 \mid N_\tau = 1] \overset{\text{def}}{=} \theta \cdot r \cdot (1-1/K) \cdot 1 - \theta \cdot r \cdot 1$$

(13)

By rearranging terms, you can see that there will be no excited speculators left whenever the asset’s past returns are sufficiently low, $r < r_* = 1/\theta$, as shown in the left half of Figure 1. However, if $r > r_*$, the exact opposite intuition will hold. If a single speculator happens to get excited about the asset, then this lone agent will likely be able to excite a friend before he himself comes to his senses. The excited-speculator population will remain above zero in the steady state as shown in the right half of Figure 1.

\(^9\) $\bar{n} \in SS(\theta,r)$ is stable if for every $\delta > 0$ there is some $\epsilon > 0$ such that $|n_\tau(n_0,\theta,r) - \bar{n}| < \delta$ for all $\tau \geq 0$ given any initial population $n_0 \in (\bar{n} - \epsilon, \bar{n} + \epsilon)$.\(^{9,1}\)
\[
\bar{n} = (r - r*) / r
\]

Figure 1. On/Off Switch. (Top) \(x\)-axis: an asset’s past return, \(r \in (0, \infty)\). \(y\)-axis: steady-state solutions, \(\bar{n} \in SS(\theta, r)\), for a population of excited speculators. Solid black line reports stable steady states; dashed red line reports unstable ones. Population displays a bifurcation at \(r* = 1/\theta\). (Bottom) Transition to steady state when \(r < r_*\) vs. when \(r > r_*\). \(x\)-axis, top: fraction of speculators who are currently excited about an asset, \(n\). \(y\)-axis, top: growth rate of excited-speculator population, \(\frac{dn}{d\tau}\). When \(r < r_*\), this growth rate is always negative for all \(n > 0\) as indicated by the solid line remaining below the \(x\)-axis. By contrast, when \(r > r_*\), this growth rate is positive for some population values, \(n > 0\), as indicated by the solid line arching above the \(x\)-axis. \(x\)-axis, bottom: time since initial group of \(n_0 \geq 0\) speculators got excited about an asset at time, \(\tau = 0\). \(y\)-axis, bottom: number of speculators excited about an asset at time \(\tau > 0\), \(n_\tau = n_\tau(n_0, \theta, r)\). Different shades of gray denote different initial population sizes, \(n_0 \in [0, 1)\). When \(r < r_*\), any initial population of speculators \(n_0 > 0\) that happens to get excited will quickly lose interest and disperse so \(n_{\tau}(n_0, \theta, r) \to \bar{n} = 0\). But, when \(r > r_*\), the excited-speculator population will converge to \(\bar{n} = (r - r_*) / r > 0\) whenever a single speculator happens to get excited.
2.3 *Ex Ante* Likelihood

I now fold the limits-of-arbitrage setup from Subsection 2.1 and the on/off switch from Subsection 2.2 into a single asset-pricing model and show how to use this model to predict the *ex ante* likelihood of speculative bubbles in a way that does not depend on the nitty gritty details of the limits-to-arbitrage model involved.

I connect the previous two subsections by assuming that the number of excited speculators in the limits-to-arbitrage model from Subsection 2.1 is given by the steady-state solution in Proposition 2.2 when $r = r_{t-1}$:

$$n_t = \begin{cases} 
(r_{t-1} - r_*)/r_{t-1} & \text{if } r_{t-1} > r_* = 1/\theta \\
0 & \text{otherwise}
\end{cases}$$

(14)

It is as if at the start of each discrete time period $t$, a single speculator gets excited about an asset. After he enters, the madness of crowds either takes over or does not depending on both the asset’s past return, $r = r_{t-1}$, and its $\theta$ parameter. The resulting steady-state excited-speculator population (if one exists at all) is responsible for any non-fundamental demand shock realized in period $t$. This model-timing assumption emphasizes the distinction between the forces that sustain a speculative bubble and those that govern its likelihood.\textsuperscript{10}

A speculative bubble occurs when newswatchers push an asset’s realized return above $r_*$, causing a crowd of excited speculators to flood the market. Let $B_t$ be an indicator variable for whether there is a non-zero population of excited speculators:

$$B_t = B(\theta, r_{t-1}) \stackrel{\text{def}}{=} 1[r_{t-1} > 1/\theta] = 1[n_t > 0]$$

(15)

When $n_t > 0$ and $B_t = 1$, Proposition 2.1 says that there will be an equilibrium pricing error due to the limits of arbitrage.

It’s easiest to understand the asset-pricing implications of this model by studying the sample price path in Figure 2. This figure represents a single

\textsuperscript{10}Previous research focused on continuous feedback between speculator demand and prices (Shiller, 1984; Shive, 2010; Burnside et al., 2016; Hirshleifer, 2020). These papers operated within the limits-to-arbitrage framework, and continuous feedback is essential for understanding how a bubble episode is sustained. But, it is not essential for predicting how often they will occur. See Corollary 2.3. It is possible to show that the model’s predictions about *ex ante* bubble likelihoods carry over to a setting with continuous feedback (see Online Appendix).
Figure 2. Sample Price Path. Single 100-period simulation using $\psi = 0$, $\mu_v = 1.0$, $\kappa_v = 0.1$, $\sigma_v = 0.1$, $\theta = 0.4$, and $\lambda = 0.5$. $x$-axis represents time, $t = 1, 2, \ldots, 100$. (Top) Black line is price level, $p_t = P_t(n_t)$. Thin green line is per-period payout, $v_t$. Red shaded regions denote times when excited speculators caused arbitrageur constraints to bind and a speculative bubble to form. (Middle) Black line is realized return in previous period, $r_{t-1} = (p_{t-1} + v_{t-1})/p_{t-2}$. Dashed blue line is threshold return level, $r_* = 1/\theta$. When $r_{t-1} < r_*$, there are no excited speculators, $n_t = 0$. (Bottom) Red vertical bars report number of excited speculators, $n_t$. Four bubbles where $B_t = 1$ are labeled $t_1$, $t_2$, $t_3$, and $t_4$.

100-period realization simulated using parameters $\psi = 0$, $\mu_v = 1.0$, $\kappa_v = 0.1$, $\sigma_v = 0.1$, $\theta = 0.4$, and $\lambda = 0.5$. The black line in the top panel depicts the equilibrium price each period, $p_t = P_t(n_t)$, while the thin green line depicts the asset’s fundamental value, $v_t$. Because the risky asset is in zero net supply for this simulation, $\psi = 0$, these two lines fall right on top of one another when there are no excited speculators in the market, $n_t = 0$. This is exactly what happens most of the time. In the middle panel, the black line represents the asset’s realized return in the previous period, $r_{t-1} = (p_{t-1} + v_{t-1})/p_{t-2}$. You can see that this value is typically below the dashed blue line representing $r_* = 1/\theta$.

However, there are four different points in time where newswatchers pushed the asset’s past return above the critical threshold level—i.e., where $B_t = B(\theta, r_{t-1}) = 1$. These instances are denoted by $t_1$, $t_2$, $t_3$, and $t_4$ on the $x$-axis in the bottom panel. The height of the red bars in the bottom panel depicts the
size of the excited-speculator population during each period, \( n_t \). The empirical analysis will investigate how often we should expect to see these episodes—e.g., should we expect four episodes or just two?

Proposition 2.2 tells us how the critical performance threshold is set, \( r_* = 1/\theta \). Proposition 2.1 tells us how newswatchers link an asset’s price level to its fundamental value during normal times, \( p_t = v_t - \gamma \times \psi \). We also know how an asset’s fundamental value fluctuates over time based on Equation (1). Thus, the model predicts how \( \theta \) affects the \emph{ex ante} likelihood of a speculative bubble. If two assets have identical fundamental parameters \((\mu_v, \kappa_v, \sigma_v)\) and past performance, \( r_{t-1} < r_* \) at time \( (t-1) \), the asset with the higher speculator-persuasiveness sensitivity, \( \theta \), is more likely to experience a speculative bubble at time \( (t+1) \).

**Proposition 2.3** (Ex Ante Likelihood). Assume \( r_{t-1} < r_* \). Controlling for fundamentals at time \( (t-1) \), the probability of a speculative bubble occurring at time \( (t+1) \) is strictly increasing in \( \theta \):

\[
\frac{\partial}{\partial \theta} E_{t-1}[ B_{t+1} | r_{t-1} < r_* ] > 0
\]  

(16)

This paper treats \( \theta \) as an exogenous asset-specific constant encapsulating all of the things that make one asset’s speculators more/less persuasive following high returns than another’s. It does not explain why different assets have different \( \theta \)s. This is an interesting topic for future research, but you do not need to know why \( \theta \) varies across assets when using \( \theta \) to predict the likelihood of future bubble episodes, and there are no other models explaining the \emph{ex ante} likelihood of bubbles. By analogy, the CAPM (Sharpe, 1964) is not a bad model because William Sharpe never explained why some stocks have higher market betas. You do not need this information to use the CAPM to explain discount rates.\(^{11}\)

A key fact about Proposition 2.3 is that it does not depend on the severity

\(^{11}\)It is also not essential that \( \theta \) be strictly constant over time. All that matters is that \( \theta \) varies much more slowly than speculative booms/busts in returns, and I verify this necessary condition empirically in Section 3. Predicting the likelihood of a future bubble means looking for some property that is a) related to bubble formation and b) the same both during and between bubble episodes. That way we can measure this property during normal times and then use it to make out-of-sample predictions. If bubbles occur in a model because some parameter just suddenly changes, we cannot use the model to predict when these sudden changes are most likely to occur.
of speculator bias, $\lambda > 0$. This captures the sense in which the details of the limits-to-arbitrage model from Subsection 2.1 are not important. ‘How?’ and ‘how often?’ are two fundamentally different kinds of questions. Discovering how Dr. Jekyll turns himself into Mr. Hyde tells you nothing about how often you should expect to find a crazed monster terrorizing the streets of Victorian London (Stevenson, 1886). Corollary 2.3 formalizes this intuition.

**Corollary 2.3 (How vs. How Often).** Assume $r_{t-1} < r_\star$. After controlling for fundamentals at time $(t - 1)$, changes in $\lambda$ do not affect the probability of a speculative bubble occurring at time $(t + 1)$:

$$\frac{\partial}{\partial \lambda} E_{t-1}[B_{t+1} \mid r_{t-1} < r_\star] = 0$$

(17)

Figure 3 illustrates the logic behind this Corollary 2.3. The left panel depicts periods $t = 40, \ldots, 80$ of the sample price path in Figure 2, which was simulated using $\lambda = 0.50$. The right panel depicts the exact same time period for the exact same simulation in a world where speculator biases are $50\%$ more extreme, $\lambda = 0.75$. Increasing $\lambda$ from 0.50 to 0.75 increases the length of the second and third bubble episodes. When $\lambda = 0.50$, the second bubble episode only lasts one period; whereas, when $\lambda = 0.75$, it lasts two. Likewise, the third bubble episode goes from three to four periods long when $\lambda$ increases from 0.50 to 0.75.

Yet, the $50\%$ increase in the severity of speculators’ bias does not affect the number of bubble episodes—i.e., it does not affect *ex ante* likelihood of a bubble. This is an important distinction because policymakers are often interested in the *ex ante* likelihood of a bubble and not the specific bias×constraint pair involved. It is common to see articles discussing whether or not “China’s stimulus program is prone to blow more bubbles in the economy next year.”12 The main concern in these news articles is the likelihood that a future speculative bubble will occur. They do not focus on the specific bias×constraint pair involved.

### 3 Empirical Support

Predicting the *ex ante* likelihood of a future bubble means finding some parameter, $\theta$, that is both related to bubble formation and relatively stable over time. That

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way, researchers can measure $\theta$ during normal times and use it to make out-of-sample predictions about the likelihood of a future bubble episode. If bubbles occur because some parameter just suddenly changes, we cannot use the model to predict when these sudden changes are most likely to occur. The model developed in the previous section claims that $\theta$ is the sensitivity of speculator persuasiveness to changes in past returns. The model predicts that speculative bubbles should occur more often in assets where small increases in past returns make excited speculators relatively more persuasive to their peers. This section empirically verifies this central prediction.

I begin in Subsection 3.1 by providing a working definition of industry-level speculative bubbles. This working definition of a bubble captures the main industry-level episodes that often get called bubbles, such as the rise and fall of technology stocks during the late 1990s and early 2000s. However, I show in
the analysis below that the model’s *ex ante* predictions about the likelihood of a speculative bubble are robust to *ex post* disagreement over how to define a bubble. In layman’s terms, if you think that the bubble definition I am using wrongly includes some market episode that was not actually a bubble, then no matter. Feel free to leave that episode out. The results will be unchanged.

Next, in Subsection 3.2, I lay out the empirical approach I use. The previous literature on the econometrics of speculative bubbles has been interested in predicting the timing of the crash; however, the current paper is interested in the *ex ante* likelihood of a future speculative bubble regardless of whether the crash will go down in history books as “Black Monday” or “Black Tuesday”. So, to avoid complications related to how each speculative bubble evolves, I use a case-control methodology. I match each bubble episode (these are the cases of interest) to another otherwise identical industry-level observation where no subsequent bubble occurs (these are the control observations) based on data observed prior to the start of each bubble episode.

The end result of this matching process is a matched dataset containing pairs of *ex ante* identical industry-level observations, one that subsequently experiences a speculative bubble and another that does not. In Subsection 3.3, I show how to create an empirical estimate for the key $\theta$ parameter for each industry using data on media coverage. It is important to emphasize that I am not just proxying for $\theta$ using total media coverage. $\theta$ is the sensitivity of speculator persuasiveness to an industry’s past returns not total coverage. Finally, in Subsection 3.4, I show that the estimated $\theta$ value predicts which of each matched pair of industry-level observations will subsequently realize a bubble even in the presence of some *ex post* disagreement over how to define a bubble.

### 3.1 Bubble Episodes

Here is the benchmark working definition of an industry-level speculative bubble that I will use in this paper. In the analysis below, however, I show that this paper’s *ex ante* predictions about bubble likelihoods are robust to some *ex post* disagreement about this working definition. Everyone does not have to agree that this is the right way to define a bubble for the results in this paper to hold.
I study value-weighted monthly returns for the 49 Fama and French (1997) industries from January 1960 to December 2017. I remove the “Other” industry because this industry designation does not represent a cohesive collection of firms, leaving industries $i = 1, \ldots, 48$. Let $\text{ret}_{i,t}$ denote the $i$th industry’s return in month $t$. When computing industry-level variables, I only include firms listed on the NYSE, AmEx, and NASDAQ exchanges with share codes 10, 11, or 12. I only include industry $\times$ month observations with at least five firms.

Many historical accounts of bubbles have a strong industry component. The DotCom bubble is a prototypical example. Relative to using national-level price indexes à la Kindleberger (1978), analyzing industry-level returns also results in higher-powered statistical tests, and statistical power is important given how rare bubbles are. Aggregating up to the industry-level also helps circumvent problems posed by the entrance of new firms in bubble industries.

Let $\text{pastRet2Yr}_{i,t}$ be the $i$th industry’s raw return over the past two years:

\[
\text{pastRet2Yr}_{i,t} \equiv \prod_{t' = t - 23}^{t} (1 + \text{ret}_{i,t'}) - 1
\]  

(18)

If $\text{pastRet2Yr} = -100\%$, you would have lost your entire $1 investment from two years ago; whereas, if $\text{pastRet2Yr} = 200\%$, your initial $1 investment would have tripled in value. Let $\text{pastRetNet}_{i,t}$ be an industry’s return net of the market over the past two years, and let $\text{pastRet5Yr}_{i,t}$ be its past five-year raw return.

I use the term ‘boom period’ or ‘boom’ to refer to a period of explosive

industry-level price growth. Specifically, an industry-level boom begins when an industry first realizes $100\%+$ returns over the past two years (raw and net) and $50\%+$ raw returns over the past five years:

$$\text{boom}_{i,t} \equiv 1\{\text{pastRet}2\text{Yr}_{i,t}>100\%\} \times 1\{\text{pastRetNet}_{i,t}>100\%\} \times 1\{\text{pastRet}5\text{Yr}_{i,t}>50\%\} \quad (19)$$

Booms continue as long as an industry’s two-year raw returns remain above $100\%$. Price booms are instances where something extreme happened, either a speculative bubble or really good news. There are 53 booms in my sample.\(^{14}\)

A ‘bubble episode’ or ‘bubble’ is a collection of one or more industry-level booms that ends in a crash. I start with the set of 53 booms. Then, I search the five years following the start of each boom for a local price maximum—i.e., a place where the industry’s price is higher than at any point during the next two years. Several different booms may share the same peak. Conversely, an industry’s price might just keep on rising following a boom. Both outcomes follow from Fama (2014)’s observation that predicting crashes is hard.

Here is how I define a crash. Conditional on a peak existing, I look for the first post-peak month that is a local price minimum—i.e., a place where the industry’s price is lower than at any point during the subsequent two years. I consider this the end of the market episode associated with the peak. The post-peak drop in an industry’s price is its return from the peak to this local minimum. A speculative bubble is any peak that is followed by a $50\%+$ price decline during the next five years. To be considered a bubble, prices in an industry need to double in under two years and then fall by more than half.

Figure 4 shows that the 23 industry-level bubble episodes in my sample. I define the start of a bubble as the first month prior to the peak in which an industry has $50\%+$ raw returns over the past two years. In the analysis below, I focus on the subset of 15 bubble episodes which have valid data for the entire 60-month window prior to the start of the bubble. To have valid data, an industry may not have missing observations or experience another bubble episode during this time period. If there was another bubble episode during the five years before, then it is not right to treat the pre-bubble data as taken during ‘normal times’.

\(^{14}\)I provide additional summary statistics and examples illustrating the working definitions of boom and bubble episodes in the Online Appendix.
Figure 4. Bubble Episodes. Each panel reports results for a different industry. $x$-axis: January 1960 to December 2017. $y$-axis (log scale): dollar value of a continuously re-invested portfolio starting with $1$ in January 1970. Black line: portfolio size in month $t$. Vertical green region: separate market episode containing at least one boom period. Right-most boundary is a local peak in an industry's price level. Left-most boundary is last month prior to the peak in which the industry had raw two-year returns below 50%. Vertical red region: maximum drop in price following the peak. Opaque red regions indicate a price drop $\geq 50\%$ (a crash). Transparent red regions indicate a price drop $< 50\%$. 

19
3.2 Case-Control Study

I use a case-control methodology to test the main prediction in Proposition 2.3. I start by gathering data on the industry×month observation immediately preceding each bubble episode. Then, I match each of the pre-bubble industry×month observations—i.e., the cases of interest—to another industry×month observation with no subsequent bubble—i.e., the control observations. The matching is done based on each industry’s performance over the past several years leading up to the bubble. So, both the pre-bubble observation and the matched observation have strong returns at the time of matching. Both observations in each pair tend to have high P/E ratios. They also both tend to involve industries with younger firms. Finally, after creating this matched dataset, I check whether the \( \theta \) observed prior to the start of each industry-level episode (see Subsection 3.3) helps predict which element of each matched pair will subsequently realize a bubble.

I match pre-bubble observations to otherwise similar industry×month observations with no subsequent bubble based on each industry’s returns over the past two years (both raw and net of the market), its raw returns over the past five years, its average P/E ratio over the past two years, and its return volatility over the past two years. Let \( \text{retVol}_{i,t} \) denote the annualized volatility of returns over the past two years. Finally, let \( \text{CAPE}_{i,t} \) denote the value-weighted average of the cyclically adjusted P/E ratio in an industry. Note that I am computing \( \text{CAPE} \) separately for each industry rather than downloading the aggregate time series from Robert Shiller’s website\(^{15}\). I want to ensure that the matched sample is being drawn from normal times, so I forbid matches to be selected from any boom period plus or minus five years.

Although this matching procedure creates a matched dataset by pairing off bubble episodes with non-bubble counterparts, it is not introducing look-ahead bias. It is doing the exact opposite. It is creating a dataset containing pairs of observations that all look like they both could be followed by a speculative bubble, but only half of them actually are. The challenge below will be to see whether \( \theta \) successfully predicts which half. What’s more, because the observations

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are matched based on data available at the start of the bubble, the timing of the crash will not affect the test. All that will matter is whether a speculative bubble subsequently occurs at all.

Suppose we decide to flip two fair coins, \( \Pr[H] = 50\% \). Prior to any coins being flipped, it would be perfectly reasonable for us to define an event that depends on both the first and second tosses. For example, among all two-toss sequences, \( \{HH, HT, TH, TT\} \), what is the probability of flipping a heads then a tails, \( HT \)? This is a perfectly reasonable question to ask. The answer is \( \Pr[HT] = 25\% \), and it is the analog of asking about the \textit{ex ante} probability of a future bubble in a matched sample of \textit{ex ante} identical industry\(\times\)month observations.

For a bubble to occur, the demand coming from a group of psychologically biased speculators must be causing rational arbitrageurs’ constraints to bind. Here are the bias proxies I use in my analysis (cf. Greenwood et al., 2018). Let \( \%\Delta sales_{i,t} \) be the value-weighted average of the annualized sales growth in an industry over the past two years. Let \( \text{turnover}_{i,t} \) be the value-weighted average of the ratio of trading volume to shares outstanding over the past two years. Let \( \%\text{issuer}_{i,t} \) be the fraction of firms that issued equity in the past year weighted by market cap. Let \( \text{retAccel}_{i,t} \equiv \prod_{t'=t-11}^{t} (1 + \text{ret}_{i,t'}) - \prod_{t'=t-23}^{t-12} (1 + \text{ret}_{i,t'}) \) be the difference between an industry’s returns last year and the year before.

Table 1 shows summary statistics for the 15 pre-bubble observations as well as the matched sample of 15 industry\(\times\)month observations with no subsequent bubbles. The 15 pre-bubble industry\(\times\)month observations look just like their matched counterparts in terms of recent performance in the first five rows. This is mechanical. The matched data point for each of the 15 pre-bubble industry\(\times\)month observations was selected based on its similarity in terms of recent performance. However, there is no reason why the pre-bubble observations and their matched counterparts need to look similar along other dimensions.

In particular, it is widely believed that new and innovative industries which receive lots of media coverage are more likely to experience a bubble in the future. There is an entire chapter in Shiller (2000) entitled “New Era Thinking”. However, while it is true that bubble industries do tend to have young innovative firms that get lots of media coverage, this is not the conditional probability we
care about. We want to know the likelihood of an industry experiencing a bubble conditional on it having these characteristics not the likelihood of an industry having these characteristics conditional on experiencing a bubble. The final four rows of Table 1 show that industries with young innovative firms and lots of media coverage are no more likely to experience future bubble episodes than industries with similar recent performance.

I use the following four variables to do demonstrate this fact. Let $\% \Delta \text{age}_{i,t}$ be the percent change in the average age of firms in an industry over the past two years weighted by market cap. Here, a firm’s age is the number of years since it first appeared in the Compustat database. Let $\text{rdToSales}_{i,t}$ be the value-weighted average of the R&D-to-sales ratio during the past two years. Let $\text{articles2Yr}_{i,t}$ be the average number of Wall Street Journal (WSJ) articles each month that reference the an industry in the title during the past two years. Let $\% \Delta \text{articles2Yr}_{i,t}$ be the percent change in the number of WSJ articles per month over that same period. I collect the data on the number of times an industry is mentioned in the title of a WSJ article each month as a proxy for the amount of media coverage an industry receives. These article counts are collected by searching WSJ Historical Archives available via ProQuest.

3.3 Estimating $\theta$

Here is how I use media coverage to estimate an empirical proxy for each industry’s $\theta$ parameter.

Think about what happens when you place a glass of water in the freezer, causing it to slowly cool from room temperature to well below freezing. Water molecules in the glass are always trying to attract one another and form ice crystals, even at room temperature. It is just that the random jiggling of heat energy tends to break up any embryonic two-molecule ice crystals faster than these couplets can attract more of their neighbors at room temperature. So, larger ice crystals cannot form, and the glass of water remains liquid. But, as the temperature steadily drops, this random jiggling gets less and less frenetic. A sudden qualitative change occurs when the temperature in the glasses crosses below the critical threshold of 0° Celsius. Below this threshold, embryonic two-
<table>
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<tr>
<th></th>
<th>Pre-Bubble Avg</th>
<th>Pre-Bubble Sd</th>
<th>Matched Avg</th>
<th>Matched Sd</th>
<th>Diff</th>
</tr>
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<td>pastRet2Yr</td>
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<td>25.83</td>
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<td>4.04</td>
<td>6.68</td>
<td>3.58</td>
<td>-0.03</td>
</tr>
<tr>
<td>%Δsales</td>
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<td>9.84</td>
<td>17.41</td>
<td>14.92</td>
<td>2.46</td>
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<tr>
<td>turnover</td>
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<td>8.09</td>
<td>9.06</td>
<td>7.05</td>
<td>5.31**</td>
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<tr>
<td>%issuer</td>
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<td>14.03</td>
<td>7.31</td>
<td>6.69**</td>
</tr>
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Table 1. Matched Dataset. Summary statistics for 15 pre-bubble industry×month observations as well as a matched sample of industry×month observations with no subsequent bubble episode. Each pre-bubble industry×month observation is matched to its nearest counterpart based on its recent performance using the variables pastRet2Yr, pastRetNet, pastRet5Yr, retVol, and CAPE. pastRet2Yr is raw returns over past two years. pastRetNet is returns over the past two years net of the market. pastRet5Yr is raw returns over the past five years. retVol is annualized return volatility. CAPE is cyclically adjusted P/E ratio. The pre-bubble and matched observations will look similar along these five dimensions by construction. However, these two samples can differ along other dimensions. %Δsales is annualized sales growth. turnover is ratio of trading volume to shares outstanding. %issuer is percent of firms that issued equity. retAccel is difference between raw returns last year and the year before. %Δage is percent change in average firm age. rdToSales is the ratio of R&D expenditures to sales. articles2Yr is average number of WSJ articles per month that reference an industry in the title. %Δarticles2Yr is growth in an industry’s WSJ mentions per month. Diff: difference between the average value in bubble episodes vs. in matched sample. Numbers in parentheses are standard errors clustered by industry. Significance: * = 10%, ** = 5%, and *** = 1%. 

23
molecule ice crystals now tend to attract neighboring molecules faster than Brownian motion can shake them apart. Larger crystals can form, and the entire glass freezes solid.

It is possible to measure how strong the attraction between molecules is in a given liquid by measuring how fast unstable two-molecule crystals break apart at various temperatures above the freezing point. Suppose you start out with two liquids that are both at 20° Celsius. Then, you drop the temperature by one degree to 19° Celsius. Neither liquid freezes at this new lower temperature. But, unstable two-molecule crystals will now take longer to break apart in both liquids because there will be less random jiggling due to heat energy. If this one degree drop in temperature has a bigger affect on the length of time that transient crystals stick around in the first liquid, then what have you learned? You have learned that the forces of attraction between molecules in the first liquid must be stronger than in the second. The freezing point of the first liquid must be higher.

Something analogous is going on with excited-speculator population dynamics during normal times. Instead of molecules trying to attract another, excited speculators are trying to attract their friends. Instead of heat energy breaking apart small crystals, common sense disperses small crowds of excited speculators. As an industry’s recent performance improves, any speculators who are excited about that industry will become more persuasive to their friends. In the same way that we can gauge the level of attraction between molecules in a particular liquid by measuring how much a change in the liquid’s temperature affects the half-life of small unstable crystals, we can gauge the persuasiveness of a particular industry’s speculators by measuring how much a change in recent performance affects the half-life of its transient speculate population.

Suppose that an infinitesimal population of speculators, \( n_0 > 0 \), gets excited about an industry at time \( \tau = 0 \). Let \( \tau_{1/2} \) denote the time it takes for half of these excited speculators to regain their senses, \( \tau_{1/2} \overset{\text{def}}{=} \min \left\{ \tau > 0 : \frac{n_0 - n}{n_0} \leq \frac{1}{2} \right\} \). Proposition 2.2 dictates that \( \tau_{1/2} < \infty \) during normal times; whereas, once a boom has begun, \( \tau_{1/2} = \infty \) since this implies that the excited-speculator population will not decay to zero:

\[
\tau_{1/2}(\theta, r) = \log(2) \cdot (1 - \theta \cdot r)
\]

(20)
Let $\epsilon$ denote a small fluctuation in past returns, $r \mapsto r_\epsilon \overset{\text{def}}{=} r \cdot (1 + \epsilon)$. e.g., in the econometric analysis below, $r$ will correspond to an industry’s return over the past two years while $\epsilon$ will correspond to an industry’s return in the most recent month. Let $\Delta \tau_{1/2}$ denote how this small $\epsilon$ change in recent performance will affect the rate at which its excited speculators calm down:

$$\Delta \tau_{1/2}(\theta, r; \epsilon) \overset{\text{def}}{=} \tau_{1/2}(\theta, r_\epsilon) - \tau_{1/2}(\theta, r) \tag{21}$$

If small increases in past returns lead to large increases in the amount of time it takes for excited speculators to calm down, the following quantity will be large:

$$\Delta \tau_{1/2}(\theta, r; \epsilon) \times \epsilon \tag{22}$$

If increasing past returns does not affect the rate at which an industry’s excited speculators calm down very much, then $\Delta \tau_{1/2}(\theta, r; \epsilon) \times \epsilon \approx 0$. So, if we had an empirical proxy for how long speculators remained excited about an industry during normal times, Proposition 3.4 tells us how to use this proxy to estimate $\theta$.

**Proposition 3.4 (Estimating $\theta$).** Suppose we regressed changes in the half-life of an industry’s excited-speculator population on small fluctuations in past returns:

$$\Delta \tau_{1/2} = \hat{a} + \hat{b} \cdot \epsilon + \hat{e} \tag{23}$$

The slope coefficient will be increasing in the value of $\theta$ for that industry, $\frac{\partial \hat{b}}{\partial \theta} > 0$.

Figure 5 depicts the intuition behind this approach. Both panels show the rate at which an initial population of excited speculators decays towards $\bar{n} = 0$ when an industry’s past return is below the critical threshold, $r < r_\star$. The gray lines represent this transition when the industry’s past return is precisely $r$; whereas, the black lines represent this transition when the industry’s past return is $r_\epsilon = r \cdot (1 + \epsilon)$ for $\epsilon > 0$ and $r_\epsilon < r_\star$. The half-life of the excited-speculator population corresponds to the point on the $x$-axis at which the intersection with the horizontal dotted line at $\frac{\bar{n}}{2}$ occurs. Comparing Panel 5a to Panel 5b shows how increasing the sensitivity of an industry’s speculators to its past returns also increases the impact of the industry’s past returns on the half-life of its excited speculators—i.e., $\Delta \tau_{1/2}(\theta, r; \epsilon)$ is larger when $\theta = 0.4$ than when $\theta = 0.2$.

I use month-to-month changes in WSJ coverage, $\% \Delta \text{articles}$, to proxy for changes in the half-life of an industry’s excited-speculator population. This approach is motivated by the fact that financial news outlets strategically choose
which industries to cover so as to maximize total readership (Mullainathan and Shleifer, 2005). Thus, if an industry starts to get more coverage while \( r < r_\star \), then it is likely that its transient population of excited speculators is remaining in the market longer.\(^{16}\)

Then, to calculate my empirical proxy for \( \theta \), \textit{theta}, I run a time-series regression of the growth in the \( i \)th industry’s media coverage in month \( t \) on its raw return in the previous month, which is the analog of \( \epsilon \) in Proposition 3.4, using the past five years of monthly data:

\[
\% \Delta \text{articles}_{i,t'} = \hat{a}_{i,(t-60,t]} + \hat{b}_{i,(t-60,t]} \cdot \text{ret}_{i,t'} + \hat{e}_{i,t',(t-60,t]}
\]

for \( t' = t - 59, \ldots, t \)  

\(^{16}\)Manela (2014) also uses media coverage as a proxy for the rate of information diffusion following FDA drug approval.
Table 2. Descriptive Statistics. Distribution of $\theta$ for 30 industry×month observations in matched sample. Numbers in parentheses are standard errors clustered by industry. Significance: $^*$ = 10%, $^{**}$ = 5%, and $^{***}$ = 1%.

<table>
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Table 2 provides descriptive statistics summarizing the distribution of $\theta$. Among the 30 industry×month observations in the matched sample, the 15 pre-bubble observations tended to have higher values of $\theta$ just as would be predicted by the model. The 2.07%-point difference in means is significant at the 1% level when using standard errors clustered by industry. I provide additional summary statistics for $\theta$ in the Online Appendix. I also cross-validate my estimates of this parameter using an alternative data source to WSJ article counts—namely, Bloomberg search intensity (Ben-Rephael et al., 2017).

3.4 Main Results

This subsection shows that $\theta$ predicts the ex ante likelihood of speculative bubbles just as suggested by the model in Section 2. It also confirms that these

\[ \theta \in (0, 1) \]

means that $\theta$ can range over the entire real number line. However, please note that Proposition 3.4 says that $\frac{\partial}{\partial \theta} [\theta] > 0$. It is about differences not about levels. So, negative values of $\theta$ do not pose a problem for the empirical analysis.
predictions are robust to some ex post disagreement over how to define a bubble.

Column (1) in Table 3 presents the main empirical result, which comes from regressing an indicator variable for the 15 pre-bubble observations in the matched dataset, willBeBubble, on \( \theta \):

\[
\text{willBeBubble}_{i,t} = \hat{a} + \hat{b} \cdot \text{theta}_{i,t} + \hat{e}_{i,t}
\]  

Among the matched sample of 30 industry×month observations described above, a 1%-point increase in theta is associated with a 13.67%-point increase in the likelihood of a particular industry subsequently realizing a speculative bubble. If you just guessed at random whether an industry×month observation was one of the 15 that subsequently realized a bubble, then you would be right 50% of the time. A 13.67%-point increase on a base of 50% is an economically large effect. Because the industry×month observations in the matched dataset have similar past returns, P/E ratios, and return volatilities, this result is not explained by pre-bubble differences in performance.

Columns (2) and (3) in Table 3 shows that this main result is stable across both the first and second half of my sample. Columns (4), (5), and (6) show that the predictive power of theta is completely separate from both the proxies for speculator bias—e.g., turnover and new share issuance—and the variables related to bubble-likelihood folk wisdom—e.g., firm age, innovativeness, and media coverage—examined earlier in the paper. In particular, I want to again emphasize that theta is not just proxying for industry×month observations with lots of media coverage. Including articles2Yr and %Δarticles2Yr as right-hand-side variables in columns (5) and (6) have virtually no effect on the estimated coefficient on theta.

For an industry to have a high theta value, it has to be the case that changes in its media coverage from month to month are closely linked to its stock market performance. An industry could be getting more media coverage for lots of different reasons. The tech industry has gotten a lot more media coverage in recent years. But, this has been primarily due to talk about privacy concerns and other negative social-network externalities. Most WSJ articles talking about tech firms today are not concerned with any recent gains or losses in the stock market. Thus, even though the technology sector is getting more media coverage,
it still has a relatively low theta. By contrast, most media coverage of the technology sector during the late 1990s was specifically about the sector’s recent stock-market performance. So, in the late 1990s, the industry had a high theta.

The model in Section 2 posits that changes in recent performance have different effects on speculator persuasiveness in different industries. The idea is that each industry is endowed with a speculator-sensitivity parameter, θ, which is relatively stable over time. A speculative bubble occurs when good news pushes an industry’s recent performance above the critical threshold, \( r^\star = 1/\theta \). It is not that θ is increasing prior to a bubble. It is that returns are increasing, and this increase results in an inflow of excited speculators for high-θ industries.

Figure 6 gives evidence suggesting that this is precisely what is going on in the data. This figure takes the 30 observations in the matched sample and plots the average theta and cumulative returns of the 15 pre-bubble industry \( \times \) month observations and their 15 matched counterparts during the three years prior to the match. Recall that the matching is done based on information observed the month prior to the start of a bubble episode, e.g., for the mid-2000s mining bubble, the matching was done based on information observed as of June 2004. So, the red line in the left panel, which shows the average theta for the 15 pre-bubble observations, uses information about theta in the mining industry from July 2001 to June 2004. The red line in the right panel, which shows the cumulative returns of those same 15 pre-bubble observations, uses information about the returns of the mining industry from July 2001 to June 2004.

We know that the pre-bubble and matched observations must have similar performance in the months prior to the match. After all, a matching industry \( \times \) month observation was selected for each pre-bubble observation based on the observation’s recent performance: pastRet2Yr, pastRetNet, pastRet5Yr, retVol, and CAPE. That is why the red and black lines in the right panel sit right on top of one another prior to the match date in the right panel. What’s more, we also know that both the pre-bubble observations and their matched counterparts should have good performance prior to the match. In the right panel, we see that all 30 industry \( \times \) month observations in the matched sample saw their price levels increase by roughly 50% over the previous three years. Since the pre-bubble
Dependent Variable: \( \text{willBeBubble} \)

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Table 3. *Ex Ante* Prediction. \( \text{willBeBubble} \): indicator variable for the 15 pre-bubble observations. Coefficient of +1 indicates a 1%-point increase in the likelihood of an industry \( \times \) month observation being followed by a bubble. Each pre-bubble observation is matched to its nearest counterpart based on recent performance: \( \text{pastRet2Yr}, \text{pastRetNet}, \text{pastRet5Yr}, \text{retVol}, \text{and CAPE} \). \( \text{pastRet2Yr} \): raw returns over past two years. \( \text{pastRetNet} \): returns over the past two years net of the market. \( \text{pastRet5Yr} \): raw returns over the past five years. \( \text{retVol} \): annualized return volatility. \( \text{CAPE} \): cyclically adjusted P/E ratio. Pre-bubble and matched observations mechanically look similar along these dimensions. \( \text{theta} \): empirical proxy for sensitivity of speculator persuasiveness to recent performance. %\( \Delta \text{sales} \): annualized sales growth. turnover: ratio of trading volume to shares outstanding. %\( \text{issuer} \): percent of firms that issued equity. retAccel: difference between raw returns last year and the year before. %\( \Delta \text{age} \): percent change in average firm age. rdToSales: ratio of R&D expenditures to sales. articles2Yr: average number of WSJ articles per month that reference an industry in the title. %\( \Delta \text{articles2Yr} \): growth in an industry’s WSJ mentions per month. Numbers in parentheses are standard errors clustered by industry. Significance: * = 10%, ** = 5%, and *** = 1%.
Figure 6. Stable Parameter. Average \( \theta \) and cumulative returns in matched dataset (pre-bubble = red; matched = black) starting three years prior to and ending six months after the match date. Shaded regions: 95% confidence intervals using standard errors clustered by industry. Cumulative ret: cumulative raw returns. \( \theta \): empirical proxy for sensitivity of speculator persuasiveness to past returns. Match date: month prior to the start of bubble episode.

observations by definition subsequently realized a bubble, we should expect these 15 industries to have even higher post-match returns as shown in the right panel.

The left panel of Figure 6 is surprising. This panel shows that not only do the 15 pre-bubble observations have higher values of \( \theta \) than their matched counterparts as documented in Table 2, but it is surprising that this gap existed three years beforehand. If you looked at the \( \theta_{i,t-35} \) for each of these industries you would find roughly the same 2%-point gap between the pre-bubble and matched subsets. Further still, the level of the \( \theta \) estimates for both these groups is roughly flat over this entire three-year pre-match time period. In short, the results in Figure 6 suggest that \( \theta \) is a relatively stable industry-level property. Some industries have higher values of \( \theta \) than others. It is the interaction of good recent performance and a high value of \( \theta \) that makes these industries more likely to experience a bubble going forward.

As highlighted in the introduction, these results are robust to \textit{ex post} disagreement about how to define a bubble. Panel (a) in Table 4 shows that the coefficient on \( \theta \) remains statistically significant and has the same economic magnitude as I change various details about the definition of a bubble. Columns
### Table 4. **Ex Post Disagreement.** Dependent variable in all regressions is \textit{willBeBubble}, an indicator variable for whether observation precedes a bubble episode. Coefficient of +1 indicates a 1%-point increase in the likelihood of an industry-month observation being followed by a bubble. **Panel (a).** Each column reports the results of a separate regression using the matched dataset formed based on different bubble definitions. \textit{theta}: empirical proxy for sensitivity of speculator persuasiveness to recent performance. **Panel (b).** Each row summarizes the results of re-estimating the specification Table 3 column (1) after omitting $k \in \{0,1,2,3,4\}$ of the 15 speculative bubbles. Resulting matched dataset contains 30 observations when $k = 0$ episodes are omitted, 28 observations when $k = 1$ episode is omitted, 26 observations when $k = 2$ episodes are omitted, and so on... The first row is the same specification as in column (1) of Table 3. #Omitted: number of bubble episodes omitted from the analysis. #Obs: number of industry-month observations in the matched dataset when omitting $k$ speculative bubbles. $\binom{15}{k}$: number of ways to omit $k$ episodes from among the 15 bubble episodes.
(1) and (2) show that \( \theta \) still has roughly the same amount of predictive power when we change the start-date cutoff from 50% to 45%+ or 55%+. Columns (3) and (4) show that \( \theta \) still has roughly the same amount of predictive power when we change the boom-size cutoff from 100% to 75%+ or 125%+. Columns (5) and (6) show that \( \theta \) still has roughly the same amount of predictive power when we change this crash-size cutoff to 45%+ or 55%+.

What’s more, if you strongly believe that one of the 15 bubble episodes used in the analysis above was not a bubble, feel free to leave it out. The results will be unchanged. There are 15 different ways to remove one bubble episode from the matched sample, and the second row in Panel (b) in Table 4 reveals that, no matter which episode we disagree about, the resulting omission will not qualitatively affect the 13.67%-point estimate for the coefficient on \( \theta \).

Walking down the rows in Panel (b) of Table 4, we see that we would have to disagree on how to classify at least 5 of the 15 bubble episodes in my main sample before seeing an insignificant coefficient estimate for \( \theta \). Moreover, while there are \( \binom{15}{5} = 3,003 \) different ways to choose 5 bubble episodes from among 15 possibilities, only 2 of these choices results in a point estimate for \( \theta \) that is statistically insignificant. We would have to disagree on a third of bubble episodes in precisely the right way for \textit{ex post} disagreement to matter.

4 Conclusion

This paper aims to expand the set of questions that economists ask about speculative bubbles. Existing research on bubbles focuses on the question of how they can be sustained in equilibrium. However, not every question about bubbles is a ‘how’ question. One of the most striking things about bubbles is how rare they are. To address this ‘how often’ question, we need to add an on/off switch to an existing limits-to-arbitrage model. This paper proposes a first model of this on/off switch, which is based on the idea that speculators “go mad in herds [but] only recover their senses slowly and one by one (Mackay, 1841)”.

Bubbles should occur more often in assets where small increases in past returns make excited speculators much more persuasiveness to their peers. I show that an empirical proxy for this persuasiveness-sensitivity parameter, \( \theta \),
predicts the \textit{ex ante} likelihood of future speculative bubbles at the industry level. This paper gives the first way to forecast the likelihood of future bubble episodes based on information observed during normal times. Empirical results demonstrate that such \textit{ex ante} predictions are possible in spite of the transparent simplicity of the model governing speculator population dynamics and the fact that economists disagree about how to define a speculative bubble \textit{ex post}.

\section*{References}


A Technical Appendix

**Proof** (Proposition 2.1). Newswatchers have demand given by $x_{j,t} = (s_{j,t} - p_t)/\gamma$. Market clearing implies that $\psi = \int_0^1 x_{j,t} \cdot dj + (\lambda \cdot r_{t-1}) \times n_t$. Since newswatcher signals are correct on average, $E[s_{j,t}] = v_t$, we can conclude that:

$$\psi = (v_t - p_t)/\gamma + (\lambda \cdot r_{t-1}) \times n_t$$

Rearranging to isolate $p_t$ on the left-hand side gives the desired result. □

**Proof** (Proposition 2.2). Suppose the excited-speculator population obeys the law of motion $G(n, \theta, r) = \theta \cdot r \cdot (1 - n) \times n - n$. We can rewrite this as

$$G(n, \theta, r) = (\theta \cdot r - 1) \times n - \theta \cdot r \times n^2$$
1. If \( r < 1/\theta \), then \((\theta \cdot r - 1) < 0\). So, the only way for the right-hand side of the above equation to equal zero when \( r < 1/\theta \) is for \( n = 0 \). Thus, when \( r < 1/\theta \), \( SS(\theta, r) = \{0\} \). This unique steady state is stable since
\[
\frac{\partial}{\partial n}[G(n, \theta, r)]_{n=0, r<1/\theta} = \theta \cdot (r - 1/\theta) - 2 \cdot \theta \cdot r \cdot 0 < 0
\]
2. If \( r > 1/\theta \), then \((\theta \cdot r - 1) > 0\). So, there are now two ways for the right-hand size of the above equation to equal zero: \( n = 0 \) and \( n = (r - 1/\theta)/r \). Thus, when \( r > 1/\theta \), \( SS(\theta, r) = \{0, (r - 1/\theta)/r\} \). And, only the strictly positive steady state is stable since
\[
\frac{\partial}{\partial n}[G(n, \theta, r)]_{n=0, r>1/\theta} = \theta \cdot (r - 1/\theta) - 2 \cdot \theta \cdot r \times 0 > 0
\]
\[
\frac{\partial}{\partial n}[G(n, \theta, r)]_{n=(r-1)/r, r>1/\theta} = \theta \cdot (r - 1/\theta) - 2 \cdot \theta \cdot r \times (r - 1/\theta)/r < 0
\]

**Proof** (Proposition 2.3). The probability of realizing a speculative bubble at time \((t + 1)\) given that \( r_{t-1} < r_* \) can be written as
\[
E_{t-1}[B(\theta, r_t) | B(\theta, r_{t-1}) = 0] = Pr_{t-1}[r_t > r_* | r_{t-1} < r_*]
\]
Given the stochastic process governing fundamentals, we know that
\[
E_{t-1}[\Delta v_t] = \kappa_v \cdot (\mu_v - v_{t-1})
\]
\[
\text{Var}_{t-1}[\Delta v_t] = \sigma_v^2
\]
Thus, given knowledge of \( v_{t-1}, p_{t-1} \) and \( r_{t-1} < r_* \), we can write the probability density function (PDF) for the price of the risky asset at time \( t \) as
\[
F_{t-1}(p) = \frac{1}{\sqrt{2\pi} \sigma_v} \cdot e^{-\frac{1}{2\sigma_v^2} (p - v_{t-1} - E_{t-1}[\Delta v_t] + \gamma\psi)^2}
\]  
(A1)
This PDF can be used to write down an integral expression for the probability of a speculative bubble at time \((t + 1)\) since \( r_t = p_t/p_{t-1} \):
\[
E_{t-1}[B(\theta, r_t) | r_{t-1} < r_*] = \int_{p_{t-1}/\theta}^{\infty} F_{t-1}(p) \cdot dp
\]  
(A2)
Notice two facts about this integral. Fact \#1: because \( F_{t-1}(p) \) is a PDF, it is a strictly positive function. Fact \#2: \( \theta \) plays no part in \( F_{t-1}(p) \) itself; it only enters into Equation (A2) as a boundary condition. Thus, increasing \( \theta \) simply increases the size of the interval over which a strictly positive function is being integrated. So, \( E_{t-1}[B(\theta, r_t) | r_{t-1} < r_*] \) must be strictly increasing in \( \theta \).

**Proof** (Corollary 2.3). This corollary follows from the fact that excited speculators’ beliefs only affect equilibrium prices during a speculative bubble—i.e., when \( n_t > 0 \). However, the likelihood of entering into a speculative bubble is based on considerations made when there are no excited speculators in the market—i.e., when \( n_t = 0 \). Thus, \( \lambda \) does not show up in either the PDF in Equation (A1) or the boundary conditions in Equation (A2).
Derivation (Equation 20). Suppose an infinitesimal population of speculators, $n_0 > 0$, gets excited about the risky asset at time $\tau = 0$. The time it takes for half of this population to lose interest, $\tau_{1/2}(\theta, r)$, can be expressed as follows:

$$\tau_{1/2}(\theta, r) = \int_{0}^{\tau_{1/2}} 1 \cdot d\tau = \int_{0}^{\tau_{1/2}} \frac{dn}{dn} \cdot d\tau = \int_{n_0}^{n_0/2} d\tau \cdot dn = \int_{n_0}^{n_0/2} G(n, \theta, r)^{-1} \cdot dn$$

Since the initial population is infinitesimal, $n_0 \approx 0$, second-order terms will have a negligible impact on excited-speculator population dynamics:

$$G(n_0, \theta, r) = (\theta \cdot r - 1) \cdot n_0 + O[n_0^2]$$

We say that $f(x) = O[x]$ as $x \to 0$ if there exists $C > 0$ such that $|f(x)| \leq C \cdot x$ for all $|x| < x_{\text{max}}$. Equation (20) follows from evaluating the integral expression for $\tau_{1/2}(\theta, r)$ using only the first-order terms in $G(n_0, \theta, r)$:

$$\tau_{1/2}(\theta, r) = \int_{n_0}^{n_0/2} (\theta \cdot r - 1)^{-1} \cdot n^{-1} \cdot dn = (\theta \cdot r - 1)^{-1} \times \int_{n_0}^{n_0/2} n^{-1} \cdot dn = (1 - \theta \cdot r)^{-1} \times \log(2)$$

Proof (Proposition 3.4). Suppose we look at small fluctuations, $\epsilon \approx 0$, in an asset’s past return, $r \mapsto r_\epsilon = r \cdot (1 + \epsilon)$. Define a measure of co-movement between $\epsilon$ and the half-life of small excited-speculator populations when $r < r_*$:

$$C(\theta, r; \epsilon) = [\tau_{1/2}(\theta, r_\epsilon) - \tau_{1/2}(\theta, r)] \times \epsilon$$

The functional form for $\tau_{1/2}(\theta, r)$ given by Equation (20) implies that for small fluctuations in returns we have that $\partial_{\theta, r}[\tau_{1/2}] = \log(2) \cdot (1 - \theta \cdot r)^{-2}$ and

$$C(\theta, r; \epsilon) = [\tau_{1/2}(\theta, r_\epsilon) - \tau_{1/2}(\theta, r)] \times \epsilon = \epsilon \cdot (\theta \cdot r) \cdot \partial_{\theta, r}[\tau_{1/2}(\theta, r)] \times \epsilon = \log(2) \cdot \epsilon^2 \cdot (1 - \theta \cdot r)^{-2} \cdot (\theta \cdot r)$$

Thus, this measure of co-movement is strictly increasing in $\theta$ when $r < 1/\theta$ since:

$$\frac{\partial}{\partial \theta}[C(\theta, r; \epsilon)] = \log(2) \cdot \epsilon^2 \cdot (1 - \theta \cdot r)^{-3} \cdot (-r) \cdot (\theta \cdot r) + \log(2) \cdot \epsilon^2 \cdot (1 - \theta \cdot r)^{-2} \cdot r$$

$$= \log(2) \cdot \epsilon^2 \cdot (1 - \theta \cdot r)^{-3} \cdot r \cdot (1 + \theta \cdot r)$$

If we assume $\epsilon \sim N(0, \omega^2)$ for some small $\omega > 0$, then the expectation of $C(\theta, r; \epsilon)$ taken with respect to these small short-run fluctuations in returns will satisfy

$$\mathbb{E}[C(\theta, r; \epsilon)] = \text{Cov}[\tau_{1/2}(\theta, r_\epsilon) - \tau_{1/2}(\theta, r), \epsilon]$$

since both $\epsilon$ and $\tau_{1/2}(\theta, r_\epsilon) - \tau_{1/2}(\theta, r)$ are mean zero ignoring higher-order terms. 

\[ \square \]